# A high resolution multiple source localization based on generalized cumulant structure (GCS) matrix

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## Abstract

This paper considers a high-resolution multiple non-stationary and non-Gaussian source localization algorithm based on the proposed generalized cumulant structure (GCS) matrix that is constructed as a weighted sum of the second and fourth order cumulants of the sensor signals. The weight determines the rank and range space of the GCS matrix, and the range space of the GCS matrix should be same to the range space of the virtual array manifold matrix to estimate the true direction of arrival (DOA)s of the sources. To estimate the weight and the DOAs of sources, a rank constrained optimization problem is formulated. The optimal solution is computationally heavy, and for this reason a suboptimal solution is considered. With the weight set to an arbitrary value, singular value decomposition on the GCS matrix is performed to determine the singular matrix associated with the null space of the virtual array response matrix, and either this singular matrix or the singular matrix obtained using only the second order (SO) statistic is used to obtain the proposed spatial spectrum. Experimental results show that the proposed algorithm performs better than the recently proposed SO cumulant based algorithm for synthetic and real speech data.

**Index Terms**: high-resolution multiple source localization, second order cumulant, fourth order cumulant, non-stationary source, non-Gaussian source and virtual array manifold matrix.

## 1. Introduction

For high resolution direction of arrival (DOA) estimation that is used for source localization in many applications involving radar, sonar, seismic, electroencephalography (EEG), magnetoencephalography (MEG) and microphone array [1, 3, 5, 7, 8, 10, 12], numerous algorithms have been proposed: maximum-likelihood (ML) based algorithms [2, 5, 9] and subspace-based algorithms such as multiple signal classification (MUSIC) [1], estimation signal parameter via a rotational invariant technique (ESPRIT) [3] and various algorithms proposed recently [11, 12, 13].

The ML-based algorithms yield optimal solutions, however, are computationally heavy since they typically require nonlinear and multidimensional optimization procedure [2, 9]. In comparison with the ML-based algorithms, the subspace-based algorithms where the directions are estimated by searching a one-dimensional spatial spectrum are much more simple. To track the DOAs of impinging sources in an on-line manner, the subspace based algorithms are considered a better choice. Most subspace based high resolution multiple source localization algorithms are applicable for over-determined and determined situations- the number of sources is less than or same as the number of sensors. But in real-world scenario, the number of sources is often larger than the number of sensors and for this situation, under-determined DOA estimation algorithms based on higher order statistics (i.e., fourth order (FO) cumulant) have been proposed assuming non-Gaussian stationary sources with virtual array manifold matrix [4] [6]. For non-stationary source, Ma *et al.* [12] recently proposed a Khatri-Rao subspace based algorithm using the second order (SO) statistic.

In this paper, a high-resolution multiple source localization algorithm based on the proposed generalized cumulant structure (GCS) matrix that is constructed as a weighted sum of the SO and FO cumulants of sensor signals, is considered for under-determined situations, assuming non-stationary and non-Gaussian sources. The weight determines the rank and range space of the GCS matrix, and the range space of the GCS matrix should be same to the range space of the virtual array manifold matrix to estimate the true DOAs of the sources. To estimate the weight and the DOAs of the sources, a rank constrained optimization problem is formulated. The optimal solution is computationally heavy, and for this reason a suboptimal solution is considered. With the weight set to an arbitrary value, singular value decomposition (SVD) is performed on the GCS matrix to determine the singular matrix associated with the null space of the virtual array manifold matrix, and either this singular matrix or the singular matrix obtained using only the SO statistic is used to obtain the proposed spatial spectrum. Experimental results show that the proposed algorithm consistently outperforms the algorithm proposed by Ma *et al.* for synthetic and real speech data.

The paper is organized as follows. Section 2 describes sensor signal model and assumptions. Section 3 explains the proposed algorithm. Section 4 evaluates the proposed algorithm and Section 5 concludes the paper.

## 2. Sensor signal model and assumptions

Let $M$ sensors be uniformly spaced $d$ distance apart with $I$ $(I > M)$ wide-band sources $\{s_i(t)|i = 1, ..., I\}$ located at distinct directions.

The received sensor signal $x_m(t)$ at the $m$th sensor is modeled as,

$$x_m(t) = \sum_{i=1}^{I} a_{mi} s_i(t - \tau_{mi}) + z_m(t),\ m = 1, ..., M \quad (1)$$

where $a_{mi}$ and $\tau_{mi}$ are the attenuation factor and time delay from the $i$th source to the $m$th sensor. Here $z_m(t)$ is the noise at the $m$th sensor. Applying short-time discrete Fourier transform (ST-DFT) to (1), we obtain

$$X_{m,k}[n] = \sum_{i=1}^{I} a_{mi} S_{i,k}[n] e^{-j(2\pi k/T)(\tau_{mi} f_s)} + Z_{m,k}[n], \quad m = 1, ..., M \quad (2)$$

where $k$, $T$ and $f_s$ denote the frequency bin, the frame length and the sampling rate. Assuming far-field scenario- the size of the sensory array aperture is much smaller compared with the distance from the sources to the array [12][13]- the time delay $\tau_{mi}$ is a function of the DOA of the $i$th source $\theta_i$ such that

$$\tau_{mi} = \frac{(m-1)d\sin\theta_i}{c} \tag{3}$$

where $c$ is the source velocity. Without loss of generality, we set $a_{mi} = 1$ for $m = 1, .., M$ and $i = 1, .., I$, following the far field assumption [12][13] and denote the array response vector of the $k$th frequency bin of the $i$th source as,

$$\mathbf{a}_k(\theta_i) = [a_{1,k}(\theta_i), a_{2,k}(\theta_i), ..., a_{M,k}(\theta_i)]^T \in \mathbb{C}^{M\times 1} \tag{4}$$

where

$$a_{m,k}(\theta_i) = e^{-j(2\pi k/T)(\frac{(m-1)d\sin\theta_i}{c} f_s)}.$$

For mathematical tractability, the following four assumptions are made.

**A1)** The sources are non-stationary and non-Gaussian.
**A2)** The sources are zero-mean and mutually uncorrelated.
**A3)** The noises are stationary, the additive white zero-mean Gaussian and spatially uncorrelated.
**A4)** The source and noise are uncorrelated.

## 3. Proposed algorithm

We define the following SO and FO cumulants of the $k$th bin of the $u$,$v$ and $w$ sensor signals respectively as

$$\mathcal{K}_X^{uv,k}[n] \quad = \quad E(X_{u,k}^*[n]X_{v,k}[n]) \tag{5}$$

and

$$\begin{aligned}
&\mathcal{K}_X^{uvww,k}[n] \\
&= E(X_{u,k}^*[n]X_{v,k}[n]X_{w,k}^*[n]X_{w,k}[n]) \\
&- E(X_{u,k}^*[n]X_{v,k}[n])E(X_{w,k}^*[n]X_{w,k}[n]) \\
&- E(X_{u,k}^*[n]X_{w,k}^*[n])E(X_{v,k}[n]X_{w,k}[n]) \\
&- E(X_{u,k}^*[n]X_{w,k}[n])E(X_{v,k}[n]X_{w,k}^*[n]) \\
&,\ 1\ \le u,\ v,\ w\ \le M
\end{aligned} \tag{6}$$

where the superscript $(\cdot)^*$ is the conjugate and $E(\cdot)$ is the statistical expectation. From the four assumptions, **A1), A2), A3)** and **A4)**, the SO cumulant and FO cumulant are given by [6] [11],

$$\begin{aligned}
&\mathcal{K}_X^{uv,k}[n] \\
&= \sum_{i=1}^{I} a_{u,k}^*(\theta_i)a_{v,k}(\theta_i)E(|S_{i,k}[n]|^2) + E(Z_{u,k}^*[n]Z_{v,k}[n])
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&\mathcal{K}_X^{uvww,k}[n] \\
&= \sum_{i=1}^{I} a_{u,k}^*(\theta_i)a_{v,k}(\theta_i)(a_{w,k}^*(\theta_i)a_{w,k}(\theta_i))\mathcal{K}_S^{iiii,k}[n] \\
&= \sum_{i=1}^{I} a_{u,k}^*(\theta_i)a_{v,k}(\theta_i)\mathcal{K}_S^{iiii,k}[n]
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&\mathcal{K}_S^{iiii,k}[n] \\
&= E(|S_{i,k}[n]|^4] - 2E(|S_{i,k}[n]|^2)E(|S_{i,k}[n]|^2) \\
&- E(S_{i,k}[n]S_{i,k}[n])E(S_{i,k}^*[n]S_{i,k}^*[n])
\end{aligned}$$

In (8), the FO cumulants of the Gaussian noises are nil such that

$$\mathcal{K}_Z^{uvww,k}[n] = 0. \tag{9}$$

We have the array response element $a_{u,k}^*(\theta_i)a_{v,k}(\theta_i)$ shared by the SO and FO cumulants of sources from (7) and (8). Using the array response element, we define the SO and FO cumulant structure matrices with the virtual array manifold matrix $\mathbf{A}_k$ respectively as

$$\mathbf{R}_{\mathbf{xx},k}^{SO} \quad = \quad \mathbf{A}_k\mathbf{P}_k + \mathbf{N}_k \tag{10}$$

and

$$\mathbf{R}_{\mathbf{xx},k}^{FO} \quad = \quad \mathbf{A}_k\mathbf{C}_k \tag{11}$$

where

$$\begin{aligned}
&\mathbf{R}_{\mathbf{xx},k}^{SO} = [\boldsymbol{\kappa}_X^{SO,k}[n_1], ..., \boldsymbol{\kappa}_X^{SO,k}[n_B]] \in \mathbb{C}^{M^2\times B}, \\
&\boldsymbol{\kappa}_X^{SO,k}[n_b] = [\mathcal{K}_X^{11,k}[n_b], ..., \mathcal{K}_X^{1M,k}[n_b], \mathcal{K}_X^{21,k}[n_b], ..., \\
&\mathcal{K}_X^{2M,k}[n_b]\cdots\mathcal{K}_X^{M1,k}[n_b], ..., \mathcal{K}_X^{MM,k}[n_b]]^T \\
&\in \mathbb{C}^{M^2\times 1},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{R}_{\mathbf{xx},k}^{FO} = [\boldsymbol{\kappa}_X^{FO,k}[n_1], ..., \boldsymbol{\kappa}_X^{FO,k}[n_B]] \in \mathbb{C}^{M^2\times B}, \\
&\boldsymbol{\kappa}_X^{FO,k}[n_b] = [\mathcal{K}_X^{11ww,k}[n_b], ..., \mathcal{K}_X^{1Mww,k}[n_b], \mathcal{K}_X^{21ww,k}[n_b] \\
&, ..., \mathcal{K}_X^{2Mww,k}[n_b]\cdots\mathcal{K}_X^{M1ww,k}[n_b], ..., \mathcal{K}_X^{MMww,k}[n_b]]^T \\
&\in \mathbb{C}^{M^2\times 1},
\end{aligned}$$

$$\mathbf{A}_k = [\mathbf{a}_k^*(\theta_1)\otimes\mathbf{a}_k(\theta_1), ..., \mathbf{a}_k^*(\theta_I)\otimes\mathbf{a}_k(\theta_I)] \in \mathbb{C}^{M^2\times I},$$

$$\mathbf{P}_k = \begin{bmatrix} E(|S_{1,k}[n_1]|^2) & \cdots & E(|S_{1,k}[n_B]|^2) \\ E(|S_{2,k}[n_1]|^2) & \cdots & E(|S_{2,k}[n_B]|^2) \\ \vdots & \cdots & \vdots \\ E(|S_{I,k}[n_1]|^2) & \cdots & E(|S_{I,k}[n_B]|^2) \end{bmatrix} \in \mathbb{R}^{I\times B},$$

and

$$\mathbf{C}_k = \begin{bmatrix} \mathcal{K}_S^{1111,k}[n_1] & \cdots & \mathcal{K}_S^{1111,k}[n_B] \\ \mathcal{K}_S^{2222,k}[n_1] & \cdots & \mathcal{K}_S^{2222,k}[n_B] \\ \vdots & \cdots & \vdots \\ \mathcal{K}_S^{IIII,k}[n_1] & \cdots & \mathcal{K}_S^{IIII,k}[n_B] \end{bmatrix} \in \mathbb{C}^{I\times B}$$

where the noise matrix $\mathbf{N}_k (\in \mathbb{C}^{M^2\times B})$ is the same as the noise matrix in [12] and can be eliminated by orthogonal complement projector easily. The mathematical operator $\otimes$ is the kronecker product and $n_b$ ($b = 1, ..., B$ and $B >> I$) is the index of the block in which the source is locally stationary. In the algorithm proposed by Ma *et al.* [12], the SVD of $\mathbf{A}_k\mathbf{P}_k$ is performed to propose a spatial spectrum, assuming if $\mathbf{P}_k$ is an $I\times B$ matrix with rank $I$, then

$$\text{rank}(\mathbf{A}_k\mathbf{P}_k) = \text{rank}(\mathbf{A}_k) = I \tag{12}$$

and

$$\mathcal{R}(\mathbf{A}_k\mathbf{P}_k) = \mathcal{R}(\mathbf{A}_k) \tag{13}$$

where rank$(\cdot)$ and $\mathcal{R}(\cdot)$ represent the rank and the range space. But if $\mathbf{P}_k$ is an $I \times B$ matrix with rank $I-\alpha$ where an integer $\alpha \;\; (0 < \alpha < I)$, then

$$\begin{aligned} \mathrm{rank}(\mathbf{A}_k\mathbf{P}_k) &\leq \min\{\mathrm{rank}(\mathbf{A}_k), \mathrm{rank}(\mathbf{P}_k)\}, \\ \mathrm{rank}(\mathbf{A}_k\mathbf{P}_k) &\leq I-\alpha. \end{aligned} \tag{14}$$

This leads the algorithm proposed by Ma *et al.* to estimate DOAs incorrectly, changing the rank and the range space of $\mathbf{A}_k\mathbf{P}_k$ different from those of $\mathbf{A}_k$. Preventing this from happening, we propose a GCS matrix $\mathbf{R}^G_{\mathbf{xx},k}(w_k)$ constructed as a weighted sum of the SO and FO cumulant structure matrices as in the noiseless case,

$$\begin{aligned} \mathbf{R}^G_{\mathbf{xx},k}(w_k) &= w_k\mathbf{R}^{SO}_{\mathbf{xx},k} + (1-w_k)\mathbf{R}^{FO}_{\mathbf{xx},k} \\ &= \mathbf{A}_k(w_k\mathbf{P}_k + (1-w_k)\mathbf{C}_k) \end{aligned} \tag{15}$$

where $w_k$ is a weight. The weight $w_k$ does not effect on the rank and range space spanned by the virtual array manifold matrix $\mathbf{A}_k$ and determines the rank and range space of the $\mathbf{R}^G_{\mathbf{xx},k}(w_k)$. To make the range space of the $\mathbf{R}^G_{\mathbf{xx},k}(w_k)$ be the same as that of the $\mathbf{A}_k$, the $w_k$ has to be estimated. To determine the $w_k$ and the DOAs of sources, a rank-constrained optimization problem with the $\mathbf{R}^G_{\mathbf{xx},k}(w_k)$ and a matrix $\mathbf{U}_k \in \mathbb{C}^{M^2\times(M^2-I)}$ is formulated as

$$\min ||\mathbf{U}^H_k\{\mathbf{R}^G_{\mathbf{xx},k}(w_k)(\mathbf{R}^G_{\mathbf{xx},k}(w_k))^H\}\mathbf{U}_k||^2_2 \tag{16}$$

$$\text{subject to} \begin{cases} \mathbf{u}^H_{i,k}\mathbf{u}_{i,k} = 1, \\ \mathbf{u}^H_{i,k}\mathbf{u}_{j,k} = 0, \; i \neq j, \\ \mathrm{rank}(\mathbf{U}_k) = M^2 - I \end{cases}$$

where the operator $||\cdot||^2_2$ is the square of 2-norm of vector, the superscript $(\cdot)^H$ is the conjugate transpose and the vector $\mathbf{u}_{i,k} \in \mathbb{C}^{M^2\times 1}$ is the $i$th column of the matrix $\mathbf{U}_k$, satisfying

$$\mathrm{rank}(\mathbf{R}^G_{\mathbf{xx},k}(w_k)) = \mathrm{rank}(\mathbf{A}_k) = I \tag{17}$$

and

$$\mathcal{R}(\mathbf{R}^G_{\mathbf{xx},k}(w_k)) = \mathcal{R}(\mathbf{A}_k). \tag{18}$$

The columns of the $\mathbf{U}_k$ span the the null space of the $\mathbf{A}_k$. The rank constrained optimization problem can be solved iteratively for the $w_k$ and the $\mathbf{U}_k$. However, this requires considerable computation that nullifies the benefit of the subspace-based algorithms that includes MUSIC, ESPRIT, the algorithm proposed by Ma *et al.* and the proposed algorithm, discussed earlier. As a quick solution, the proposed algorithm sets an arbitrary value for $w_k = \hat{w}_k$ in the range $0 < \hat{w}_k < 1$ and then performs SVD on the $\mathbf{R}^G_{\mathbf{xx},k}(w_k = \hat{w}_k)$. For $\hat{w}_k$, if the $I$th largest singular value of the $\mathbf{R}^G_{\mathbf{xx},k}(w_k = \hat{w}_k)$ is larger than that of the $\mathbf{R}^G_{\mathbf{xx},k}(w_k = 1)$, we take the GCS matrix as

$$\mathbf{R}^G_{\mathbf{xx},k} = \mathbf{R}^G_{\mathbf{xx},k}(w_k = \hat{w}_k) \tag{19}$$

, otherwise

$$\mathbf{R}^G_{\mathbf{xx},k} = \mathbf{R}^G_{\mathbf{xx},k}(w_k = 1). \tag{20}$$

This procedure means that as the $I$th largest singular value increases, the range space of the GCS matrix spans the range space of the virtual array manifold matrix $\mathbf{A}_k$ more explicitly and makes the performance of proposed algorithm be the performance of the algorithm proposed by Ma *et al.* in the worst case that happens due to the bad choice of $\hat{w}_k$. Finally given

$$\mathbf{R}^G_{\mathbf{xx},k} = [\hat{\mathbf{U}}_{s,k}\hat{\mathbf{U}}_{n,k}] \begin{bmatrix} \hat{\Sigma}_{s,k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}^H_{s,k} \\ \hat{\mathbf{V}}^H_{n,k} \end{bmatrix} \tag{21}$$

where $\hat{\mathbf{U}}_{s,k} \in \mathbb{C}^{M^2\times I}$ and $\hat{\mathbf{V}}_{s,k} \in \mathbb{C}^{B\times I}$ are the left and right singular matrices corresponding to the nonzero singular values, respectively, the matrix $\hat{\mathbf{U}}_{n,k} \in \mathbb{C}^{M^2\times(M^2-I)}$ and $\hat{\mathbf{V}}_{n,k} \in \mathbb{C}^{B\times(M^2-I)}$ are the left and right singular matrices corresponding to the zero singular values, respectively, and $\hat{\Sigma}_{s,k} \in \mathbb{R}^{I\times I}$ is a diagonal matrix whose diagonals contain the nonzero singular values, the proposed spatial spectrum is given by,

$$P_{\mathrm{GCS}}(\theta) = \frac{1}{\sum_k(\mathbf{a}^*_k(\theta)\otimes\mathbf{a}_k(\theta))^H\hat{\mathbf{U}}_{n,k}\hat{\mathbf{U}}^H_{n,k}(\mathbf{a}^*_k(\theta)\otimes\mathbf{a}_k(\theta))}. \tag{22}$$

Given $\hat{w}_k$, the matrix $\hat{\mathbf{U}}_{n,k}$ is an optimal $\mathbf{U}_k$ of (16) whose columns span the null space of the GCS matrix. Notice that the proposed algorithm can be performed, reducing the dimension of the GCS matrix as the algorithm proposed by Ma *et al.*.

## 4. Performance evaluation

In this section, the proposed algorithm is evaluated for both synthetic and real speech ( non-stationary and non-Gaussian source ) data. For synthetic data, three microphones signals are synthesized using four speech signals with the true DOAs equal to $30°$, $70°$, $110°$ and $150°$ and adding white Gaussian noise. The following performance of the proposed algorithm is an average performance evaluated by various $\hat{w}_k, \;\; 0 < \hat{w}_k < 1$. Figure 1 shows the root mean squared error (RMSE) versus signal-to-noise ratio (SNR)s for the algorithm proposed by Ma *et al.* ( the SOS, as a type of MUSIC ) and the proposed algorithm ( the proposed ) on 100 Monte Carlo trials. The RMSE and SNR are defined respectively as RMSE $= \sqrt{\frac{1}{I}\sum_{i=1}^{I}E(|\theta_i - \hat{\theta}_i|^2)}$ and SNR $= 10\log_{10}(\frac{P_{x_m}}{\sigma^2_m})$ where $\theta_i$, $\hat{\theta}_i$, $P_{x_m}$ and $\sigma^2_m$ denote the true DOAs, estimated DOAs, the power of a clean microphone signal ( without noise ), $E(|x_m(t)|^2)$ and the power of white Gaussian noise, $E(|z_m(t)|^2)$ respectively. The powers of the signals are the same i.e., $P_{x_1} = P_{x_2} = P_{x_3}$ and the powers of the noises are the same i.e., $\sigma^2_1 = \sigma^2_2 = \sigma^2_3$. We demonstrate that the RMSE of the proposed algorithm is lower than that of the algorithm proposed by Ma *et al.*. For real speech data, we test the proposed algorithm using the data used in [13]. We use three microphones signals for four speech signals DOAs estimation. The true DOAs are $30°$, $70°$, $110°$ and $150°$. Figure 2 shows that the proposed algorithm ( the proposed ) and the spatial spectra of the algorithm proposed by Ma *et al.* ( the SOS ). Table 1 shows the experimental results of the DOAs accuracy for each period of the real speech data; the proposed ( the SOS ). The letter X means that there is no local peak. We can see the explicit local peaks in the circle area in Figure 2 and less letter Xs for the proposed algorithm in Table 1. Therefore, the proposed algorithm has higher DOA resolution than the algorithm proposed by Ma *et al.*.

Table 1: The experimental results of the DOAs accuracy for each period of the real speech data; the proposed ( the SOS ).

| | true $\theta$ | 0∼1 s | 1∼2 s | 2∼3 s | 3∼4 s | 4∼5 s | 5∼6 s | 6∼7 s |
|---|---|---|---|---|---|---|---|---|
| source a | 150° | 155°(158°) | 165°(165°) | 166°(166°) | 160°(168°) | 160°(160°) | 147°(X) | 146°(X) |
| source b | 110° | 110°(112°) | 112°(112°) | 110°(110°) | 117°(117°) | 109°(109°) | 109°(109°) | 110°(110°) |
| source c | 70° | 70°(70°) | 73°(73°) | 68°(68°) | 73°(73°) | 72°(72°) | 72°(72°) | 71°(71°) |
| source d | 30° | 18°(X) | 41°(41°) | X(X) | 24°(24°) | 24°(24°) | 24°(21°) | 28°(23°) |

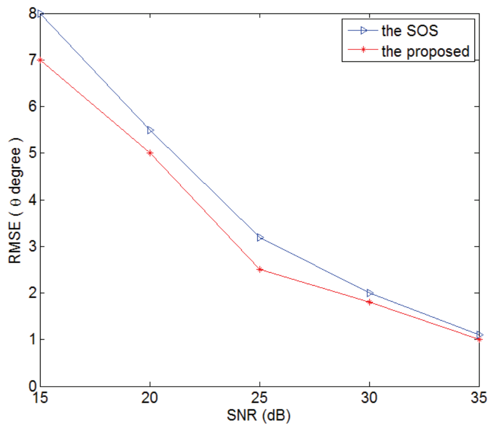


Figure 1: The root mean squared error (RMSE) versus signal-to-noise ratio (SNR)s for the algorithm proposed by Ma *et al.* ( the SOS, as a type of MUSIC ) and the proposed algorithm ( the proposed ) on 100 Monte Carlo trials.

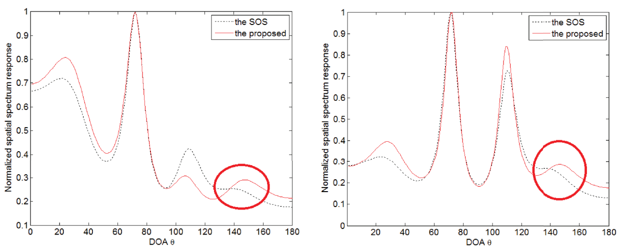


Figure 2: The normalized spatial spectra of the algorithm proposed by Ma *et al.* ( the SOS ) and the proposed algorithm ( the proposed ).

# 5. Conclusion

In this paper, a high-resolution multiple source localization algorithm based on the proposed GCS matrix that is constructed as a weighted sum of the SO and FO cumulants of sensor signals is proposed for under-determined situation. The sources are assumed non-stationary and non-Gaussian source. In experimental results, the proposed algorithm consistently outperforms the recently proposed SO cumulant based algorithm for synthetic and real speech data.

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